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Gauge/gravity correspondence from open/closed string duality

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Abstract: We compute the annulus diagram corresponding to the interaction of a fractional D3 brane with a gauge field on its world-volume and a stack of N fractional D3 branes on the orbifolds $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. We show that its logarithmic divergence can be equivalently understood as due either to massless open string states circulating in the loop or to massless closed string states exchanged between two boundary states. This follows from the fact that, under open/closed string duality, massless states in the open and closed string channels are matched into each other without mixing with massive states. This explains why the perturbative properties of many gauge theories living on the world-volume of less supersymmetric and nonconformal branes have been recently obtained from their corresponding supergravity solution.

Keywords: D-branes, String Duality.

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1. Introduction

A D brane has the twofold property of being a solution of the low-energy string effective action, which is just given by supergravity, and of having open strings with their endpoints attached to its world-volume. In particular, the lightest open string excitations correspond to a gauge field and its supersymmetric partners if the theory is supersymmetric. These complementary descriptions open the way to study quantum properties of the gauge theory living in the world-volume of a D brane from the classical dynamics of the brane and viceversa. This goes under the name of gauge/gravity correspondence that has allowed to derive properties of $\mathcal{N} = 4$ super Yang-Mills — as one can see for example in ref. [1] — and also, by the addition of a decoupling limit, to formulate the Maldacena conjecture of the equivalence between $\mathcal{N} = 4$ super Yang-Mills and type-IIB string theory compactified on $AdS_5 \otimes S_5$ [2].

Although it has not been possible to extend the Maldacena conjecture to non-conformal and less supersymmetric gauge theories, a lot of a priori unexpected information on these theories has been obtained from the gauge/gravity correspondence.[1] In particular, it has been shown that the classical supergravity solutions corresponding to fractional and wrapped D branes encode perturbative properties of non conformal and less supersymmetric gauge theories living on their world-volume, such as the chiral and scale anomaly [7, 8]. It was of course expected that those properties could be derived by studying, in string theory, the gauge theory living on the above D branes by taking the field theory limit of the one-loop open string annulus diagram, with such methods as those described for

[1] For general reviews on various approaches see for instance refs. [3]–[6].

instance in ref. [9]. But it came as a surprise that these properties were also encoded in the supergravity solution, especially after the formulation of the Maldacena conjecture, which limited the validity of the supergravity approximation to the strong coupling regime of the gauge theory.

Being gauge theories and supergravity related to open and closed strings respectively, which in turn are connected by the open/closed string duality, there must be a relation between this latter and the gauge/gravity correspondence.

In this paper we will use fractional branes to analyze features of the relation between the open/closed string duality and the gauge/gravity correspondence more deeply. In particular, we will show that, working in the pure string framework and using only the open/closed string duality, the perturbative properties of the gauge theory living on a D brane can be equivalently derived by performing the field theory limit either in the open string channel, as expected, or in the closed string channel, where the supergravity approximation holds.

We start by computing the one-loop open string annulus diagram which describes the interaction of a fractional brane of the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ having an SU(N) gauge field on it with N fractional branes without any gauge field, and we extract from it the coefficient of the gauge kinetic term. It turns out, as observed in refs. [10]–[13], that this contribution is logarithmically divergent already at the string level. In general divergences higher than the logarithmic ones correspond to gauge anomalies that make the theory inconsistent and therefore must be cancelled. Logarithmic divergences correspond instead to the exchange of massless closed string states in the closed string channel, as it can be seen from the fact that the boundary state has a non-vanishing coupling to them (*massless tadpoles*). Those divergences were originally found in the bosonic string where they were caused by the dilaton exchange in the closed string channel and were cured in different ways [14]–[20]. In the case we are examining in this paper they are due to the exchange, in the two directions transverse to the branes and the orbifold, of massless twisted states that have a nonzero coupling to fractional D3 branes and therefore contribute in the closed string channel. In general the presence of tadpoles signal some kind of instability that must be cured. On the other hand, it turns out that those logarithmic divergences precisely correspond to the one-loop divergences that one finds in the gauge theory living in the world-volume of the brane. In this paper we just regularize the string calculation introducing an infrared cutoff in the closed string channel corresponding to an ultraviolet cutoff in the open string channel and we show that the divergent contribution can be seen to come either from the exchange of massless closed string states between the two fractional branes or from the massless open string states that go around the loop. Indeed, under the modular transformation, which maps the open to the closed string channel, *open string massless states go into closed string massless states, and open string massive states go into closed string massive states, without, surprisingly, showing any mixing between massless and massive states.* By adding to the one-loop open string diagram also the contribution of the open string tree diagrams, we find an expression for the gauge coupling constant that gives the correct beta-function of $\mathcal{N} = 2$ SYM, exactly reproducing what has been found from supergravity calculations [21]–[25].

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Another important feature of our calculation is the appearance of a non vanishing contribution from the Ramond odd spin structure at the string level, giving the vacuum angle $\theta_{\rm YM}$ of the gauge theory living on the brane. The introduction of a complex cutoff together with a symmetrization between the two fractional branes occurring in the annulus diagram allows us to reproduce the corresponding supergravity calculation.

The results obtained in the case of the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ are then shown to hold also in the case of the orbifold $\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$, where the world-volume gauge theory is $\mathcal{N}=1$ SYM.

In conclusion, in this paper we show why supergravity calculations reproduce the perturbative behavior of the gauge theory living on the fractional branes of the orbifolds $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$. This is a consequence of the fact that, when we introduce an external field, the annulus diagram is divergent already at the string level and of the open/closed string duality.

What remains still a bit obscure is why the logarithmic divergence that appears *at string level* is directly related to the divergence that one finds in the gauge theory living on the brane, reproducing the correct field theory anomalies.

The paper is organized as follows. Section 2 is devoted to the calculation in the full string theory of the annulus diagram for fractional branes of the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ as a function of the gauge field on the dressed brane and of the distance between the latter and the stack of the N branes. In section (3) the field theory limit is performed both in the open and closed string channel and it is shown why the perturbative properties of the gauge theory living on N fractional D3 branes can be derived from their supergravity solution. In section (4) we extend the previous analysis to the orbifold $\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$ finding again agreement with the results obtained from the supergravity solution for $\mathcal{N}=1$ supersymmetric gauge theories. In appendix A we list the f- and Θ-functions and their transformation properties under the modular transformation. Appendix B is devoted to the calculation of the annulus diagram in the open string channel, while that in the closed string channel is performed in appendix C.

2. Branes in external fields: $\mathcal{N}=2$ orbifold

In this section we analyze the interaction between a D3 brane with an external SU(N) gauge field on its world-volume (in the following named *dressed* brane) and a stack of N ordinary D3 branes. This can be equivalently done either, in the open string channel, by computing the one-loop open string diagram or, in the closed string channel, by computing the tree closed string diagram containing two boundary states and a closed string propagator.

In order to obtain a gauge theory with reduced supersymmetry we consider type-IIB superstring theory on an orbifold space. Furthermore, to obtain a non conformal gauge theory, we study fractional D3 branes[2] which are characterized by their being stuck at the orbifold fixed point and which, unlike bulk branes, have a non-conformal theory on their world-volume. For the sake of simplicity we consider fractional branes of the orbifold $\mathbb{R}^{1,5}\times\mathbb{C}^2/\mathbb{Z}_2$ that have $\mathcal{N}=2$ super Yang-Mills on their world-volume.

[2]For a review of fractional branes and their application to the study of non-conformal gauge theories see for instance ref. [3].

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The $\mathbb{Z}_2$ group, that is chosen to be acting on the coordinates x^m with $m = 6, 7, 8, 9$, is characterized by two elements (e, h), being e the identity element and h such that $h^2 = e$. The element h acts on the complex combinations $\vec{z} = (z^1, z^2)$, where $z^1 = x^6 + ix^7$, $z^2 = x^8 + ix^9$ as follows:

$$(z_1\,, z_2) \to (-z_1\,, -z_2)\,. \tag{2.1}$$

The orbifold group Z_2 acts also on the Chan-Paton factors located at the endpoints of the open string stretched between the branes. Fractional branes are defined as branes for which such factors transform according to irreducible representations of the orbifold group and we consider only the trivial one corresponding to a particular kind of fractional branes. The orbifold we are considering is non compact and has therefore only one fixed point located at $z_1 = z_2 = 0$. We are interested in the case of parallel fractional D3 branes with their world-volume along the directions x^0, x^1, x^2, x^3, that are completely external to the space on which the orbifold acts. The gauge field lives on the four-dimensional world-volume of the fractional D3 brane and can be chosen to have the following form:

$$\hat{F}_{\alpha\beta} \equiv 2\pi\alpha' F_{\alpha\beta} = \begin{pmatrix} 0 & f & 0 & 0 \\ -f & 0 & 0 & 0 \\ 0 & 0 & 0 & g \\ 0 & 0 & -g & 0 \end{pmatrix}\,. \tag{2.2}$$

The interaction between two branes is given by the vacuum fluctuation of an open string that is stretched between them. In particular, the free energy of an open string between a D3 brane and a stack of N D3 branes located at a distance y in the plane (x^4, x^5) orthogonal to both the world-volume of the D3 branes and the four-dimensional space on which the orbifold acts, is given by the one-loop open string free energy:

$$\begin{aligned} Z &= N \int_0^\infty \frac{d\tau}{\tau} Tr_{\mathrm{NS-R}} \left[\left(\frac{e+h}{2} \right) (-1)^{F_s} (-1)^{G_{bc}} \left(\frac{(-1)^{G_{\beta\gamma}} + (-1)^F}{2} \right) e^{-2\pi\tau L_0} \right] \\ &\equiv Z_e^o + Z_h^o \end{aligned} \tag{2.3}$$

where G_{bc} and $G_{\beta\gamma}$ are, respectively, the ghost and superghost number operators, F_s is the space-time fermion number, $(-1)^F$ and L_0 are defined in appendix B, and the superscript o stands for *open*. The fact that we are considering a string theory on the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ is encoded in the presence of the orbifold projector $P = (e+h)/2$ in the trace. The explicit computation, shown in detail in appendix B, gives the following results:

$$\begin{aligned} Z_e^o &= -\frac{N}{(8\pi^2\alpha')^2} \int d^4x \sqrt{-\det(\eta + \hat{F})} \int_0^\infty \frac{d\tau}{\tau} e^{-\frac{y^2\tau}{2\pi\alpha'}} \frac{\sin\pi\nu_f \sin\pi\nu_g}{f_1^4(e^{-\pi\tau})\Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)} \times \\ &\quad \times \Big[f_3^4(e^{-\pi\tau})\Theta_3(i\nu_f\tau|i\tau)\Theta_3(i\nu_g\tau|i\tau) - f_4^4(e^{-\pi\tau})\Theta_4(i\nu_f\tau|i\tau)\Theta_4(i\nu_g\tau|i\tau) - \\ &\qquad - f_2^4(e^{-\pi\tau})\Theta_2(i\nu_f\tau|i\tau)\Theta_2(i\nu_g\tau|i\tau) \Big], \end{aligned} \tag{2.4}$$

$$Z_h^o = \frac{N}{(8\pi^2\alpha')^2} \int d^4x \sqrt{-\det(\eta + \hat{F})} \int_0^\infty \frac{d\tau}{\tau} e^{-\frac{y^2\tau}{2\pi\alpha'}} \frac{4\,\sin\pi\nu_f \sin\pi\nu_g}{\Theta_2^2(0|i\tau)\Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)} \times$$

$$\times \left[\Theta_3^2(0|i\tau)\Theta_4(i\nu_f\tau|i\tau)\Theta_4(i\nu_g\tau|i\tau) - \Theta_4^2(0|i\tau)\Theta_3(i\nu_f\tau|i\tau)\Theta_3(i\nu_g\tau|i\tau)\right] +$$
$$+\frac{iN}{32\pi^2}\int d^4x\, F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\int_0^\infty \frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}\,, \tag{2.5}$$

where $\tilde{F}_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}$. The f- and the Θ-functions are listed in appendix A. In the previous equations we have defined:

$$\nu_f \equiv \frac{1}{2\pi i}\log\frac{1+f}{1-f} \qquad \text{and} \qquad \nu_g \equiv \frac{1}{2\pi i}\log\frac{1-ig}{1+ig}\,. \tag{2.6}$$

The three terms in eq. (2.4) come from the NS, NS$(-1)^F$, and R sectors respectively, while the contribution from the R$(-1)^F$ sector vanishes. In eq. (2.5) the three terms come from the NS$(-1)^F$, NS and R$(-1)^F$ sectors respectively, while the R contribution vanishes because the projector h annihilates the Ramond vacuum.

The above computation can also be performed in the *closed string channel* where Z^c_e and Z^c_h now read as the tree level closed string amplitude between two untwisted and two twisted boundary states respectively, with the results:

$$Z^c_e = \frac{\alpha'\pi N}{2}\int_0^{+\infty} dt \quad {}^U\langle D3;F|e^{-\pi t(L_0+\bar{L}_0-a_c)}|D3\rangle^U\,, \tag{2.7}$$
$$Z^c_h = \frac{\alpha'\pi N}{2}\int_0^{+\infty} dt \quad {}^T\langle D3;F|e^{-\pi t(L_0+\bar{L}_0)}|D3\rangle^T \tag{2.8}$$

where $|D3;F\rangle$ is the boundary state dressed with the gauge field F. The details of this calculation are presented in appendix C. Here we give only the final results, i.e.:

$$Z^c_e = \frac{N}{(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^\infty \frac{dt}{t^3}e^{-\frac{y^2}{2\pi\alpha' t}}\frac{\sin\pi\nu_f\sin\pi\nu_g}{\Theta_1(\nu_f|it)\Theta_1(\nu_g|it)f_1^4(e^{-\pi t})}\times$$
$$\times\Big\{f_3^4(e^{-\pi t})\Theta_3(\nu_f|it)\Theta_3(\nu_g|it) - f_4^4(e^{-\pi t})\Theta_4(\nu_f|it)\Theta_4(\nu_g|it) -$$
$$-\Theta_2(\nu_f|it)\Theta_2(\nu_g|it)f_2^4(e^{-\pi t})\Big\}\,, \tag{2.9}$$
$$Z^c_h = \frac{N}{(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^\infty \frac{dt}{t}e^{-\frac{y^2}{2\pi\alpha' t}}\frac{4\,\sin\pi\nu_f\sin\pi\nu_g}{\Theta_4^2(0|it)\Theta_1(\nu_f|it)\Theta_1(\nu_g|it)}\times$$
$$\times\left\{\Theta_2^2(0|it)\Theta_3(\nu_f|it)\Theta_3(\nu_g|it) - \Theta_3^2(0|it)\Theta_2(\nu_f|it)\Theta_2(\nu_g|it)\right\} +$$
$$+\frac{iN}{32\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\alpha\beta}\int\frac{dt}{t}e^{-\frac{y^2}{2\pi\alpha' t}}. \tag{2.10}$$

The three terms in eq. (2.9) come from the NS-NS, NS-NS$(-1)^F$ and R-R sectors respectively, while those in eq. (2.10) from the NS-NS, R-R and R-R$(-1)^F$ ones. In particular, the twisted odd R-R$(-1)^F$ spin structure gets a non vanishing contribution only from the zero modes, as explicitly shown in appendix C.

It goes without saying that the two expressions for Z separately obtained in the open and the closed string channels are as expected, equal to each other. This equality goes under the name of open/closed string duality and can be easily shown by using how the Θ functions transform (see eq. (A.3)) under the modular transformation that relates the modular parameters in the open and closed string channels, namely $\tau = 1/t$. It can be

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easily seen that, in going from the open (closed) to the closed (open) string channel, we have the following correspondence between the various non vanishing spin structures [26]:

$$\begin{aligned} \mathrm{NS} &\leftrightarrow \mathrm{NS}-\mathrm{NS}\,, \qquad \mathrm{NS}(-1)^{\mathrm{F}} \leftrightarrow \mathrm{R}-\mathrm{R} \\ R &\leftrightarrow \mathrm{NS}-\mathrm{NS}(-1)^{\mathrm{F}}\,, \quad \mathrm{R}(-1)^{\mathrm{F}} \leftrightarrow \mathrm{R}-\mathrm{R}(-1)^{\mathrm{F}}\,. \end{aligned} \tag{2.11}$$

It is also easy to see that the distance y between the dressed D3 brane and the stack of the N D3 branes makes the integral in eq. (2.10) convergent for small values of t, while in the limit $t \to \infty$ the integral is logarithmically divergent. This divergence is due to a twisted tadpole corresponding to the exchange of massless closed string states between the two boundary states in eq. (2.8). We would like to stress that the presence of the gauge field makes the divergence to appear already *at the string level*, before any field theory limit ($\alpha' \to 0$) is performed. When F vanishes, the divergence is eliminated by the integrand being identically zero as a consequence of the fact that fractional branes are BPS states.

As observed in refs. [10, 11, 12, 13, 19] tadpole divergences correspond in general to the presence of gauge anomalies, which make the gauge theory inconsistent and must be eliminated by drastically modifying the theory or by fixing particular values of parameters. For instance, in type-I superstring they are eliminated by fixing the gauge group to be SO(32) [19]. Instead, as stressed in refs. [10]–[13], logarithmic tadpole divergences do not correspond to gauge anomalies. In the bosonic string they have been cured in a variety of ways [14]–[20]. It will turn out in the next section that, in our case, the logarithmically divergent tadpoles correspond to the fact that the gauge theory living on the brane is not conformally invariant. In fact, they provide the correct one-loop running coupling constant.

In this paper, following the suggestion of refs. [10, 11], we cure these divergences just by introducing in eq. (2.10) an infrared cutoff that regularizes the contribution of the massless closed string states. Since, in the open/closed string duality, an infrared divergence in the closed string channel corresponds to an ultraviolet divergence in the open string channel, it is easy to see that the expression in eq. (2.5) is divergent for small values of τ and needs an ultraviolet cutoff. It will turn out that this divergence is exactly the one-loop divergence that one gets in $\mathcal{N}=2$ super Yang-Mills, which is the gauge theory living in the world-volume of the fractional D3 brane.

Our results are consistent with those of ref. [27] where, in the context of unoriented theories, it is shown that string amplitudes are in general affected by both ultraviolet and infrared divergences. In our case, the finite parameter y provides a natural IR (UV) cut-off in the open (closed) channel, leaving the amplitude to be divergent only in the UV (IR) corner. Following their procedure, we could as well introduce two stringy β-functions, one for each channel, and show that they coincide. However we are interested in establishing a deeper connection with the gauge/gravity correspondence and therefore in this paper we focus on the behavior of massless and massive states separately with respect to the open/closed duality.

To this end, starting from eqs. (2.5) and (2.10), containing arbitrary powers of the gauge field F, we extract the quadratic term in the gauge field F from the previous general expressions. In the open string channel it is given by:

$$Z_h^o(F)=\left[-\frac{1}{4}\int d^4x F^a_{\alpha\beta}F^{a\,\alpha\beta}\right]\left\{-\frac{N}{8\pi^2}\int_0^\infty\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}\left[\frac{f_3(k)f_4(k)}{f_1(k)f_2(k)}\right]^4 2k\frac{d}{dk}\log\left[\frac{f_3(k)}{f_4(k)}\right]\right\}+$$
$$+iN\left[\frac{1}{32\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\right]\int_0^\infty\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}\,,\qquad k=e^{-\pi\tau}\,, \tag{2.12}$$

while in the closed string channel it reads:

$$Z_h^c(F)=\left[-\frac{1}{4}\int d^4x F^a_{\alpha\beta}F^{a\,\alpha\beta}\right]\left\{\frac{N}{8\pi^2}\int_0^\infty\frac{dt}{t}e^{-\frac{y^2}{2\pi\alpha' t}}\left[\frac{f_3(q)f_2(q)}{f_1(q)f_4(q)}\right]^4 2q\frac{d}{dq}\log\left[\frac{f_3(q)}{f_2(q)}\right]\right\}+$$
$$+iN\left[\frac{1}{32\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\right]\int_0^\infty\frac{dt}{t}e^{-\frac{y^2}{2\pi\alpha' t}}\,,\qquad q=e^{-\pi t}\,. \tag{2.13}$$

The two previous equations are derived in detail in appendices B and C respectively, and are equal to each other as one can see by performing the modular transformation $\tau = 1/t$. It can also be seen that the untwisted sector does not produce any quadratic term in the gauge field, in analogy with what happens in flat space because of its non-renormalization in $\mathcal{N}=4$ super Yang-Mills.[3]

As we have already discussed, it turns out that the divergence that we have at the string level is exactly the one-loop divergence present in the gauge field theory living in the world-volume of the brane and can be equivalently seen as due, in the open string channel, to the massless open string states circulating in the loop and, in the closed string channel, to the massless closed string states exchanged between two boundary states. This can be seen by isolating the contribution of the massless open and closed string states, respectively, in eqs. (2.12) and (2.13), which are the only ones giving a divergence. In this way in the open string channel we get:

$$Z_h^o(F)\to\left[-\frac{1}{4}\int d^4x F^a_{\alpha\beta}F^{a\,\alpha\beta}\right]\times$$
$$\times\left\{\frac{1}{g^2_{\rm YM}(\Lambda)}-\frac{N}{8\pi^2}\int_{\frac{1}{\alpha'\Lambda^2}}^\infty\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}+\frac{N}{8\pi^2}\int_0^\infty\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}G(k)\right\}+$$
$$+iN\left[\frac{1}{32\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\right]\int_{\frac{1}{\alpha'\Lambda^2}}^\infty\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}} \tag{2.14}$$

where

$$G(k)=-\left[\frac{f_3(k)f_4(k)}{f_1(k)f_2(k)}\right]^4 2k\frac{d}{dk}\log\left[\frac{f_3(k)}{f_4(k)}\right]+1\,, \tag{2.15}$$

while in the closed string channel we get:

$$Z_h^c(F)\to\left[-\frac{1}{4}\int d^4x F^a_{\alpha\beta}F^{a\,\alpha\beta}\right]\times$$

[3]For details see for instance ref. [28] where also the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ is considered in the compactification of type-I superstring.

$$\times\left\{\frac{1}{g^2_{\rm YM}(\Lambda)} - \frac{N}{8\pi^2}\int_0^{\alpha'\Lambda^2}\frac{dt}{t}e^{-\frac{y^2}{2\pi\alpha' t}} + \frac{N}{8\pi^2}\int_0^{\infty}\frac{dt}{t}e^{-\frac{y^2}{2\pi\alpha' t}}F(q)\right\} +$$

$$+iN\left[\frac{1}{32\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\right]\int_0^{\alpha'\Lambda^2}\frac{dt}{t}e^{-\frac{y^2}{2\pi\alpha' t}} \tag{2.16}$$

where

$$F(q) = \left[\frac{f_3(q)f_2(q)}{f_1(q)f_4(q)}\right]^4 2q\frac{d}{dq}\log\left[\frac{f_3(q)}{f_2(q)}\right] + 1\,. \tag{2.17}$$

Notice that in the two previous equations we have also added the contribution coming from the tree diagrams that contain the bare coupling constant. In an ultraviolet finite theory such as string theory we should not deal with a bare and a renormalized coupling. On the other hand, we have already discussed the fact that the introduction of a gauge field produces a string amplitude that is divergent already at the string level and that therefore must be regularized with the introduction of a cutoff.

We have already mentioned that eqs. (2.14) and (2.16) are equal to each other as one can see by performing the modular transformation $\tau = 1/t$. Actually, by a closer look one can see that the contribution of the massless states and the one of the massive states transform respectively into each other without any mixing between massless and massive states. Indeed the contribution of the massless closed string states can be easily obtained by performing the modular transformation on that of the massless open string states and viceversa. Furthermore the threshold corrections, corresponding to the contribution of the massive states in the two channels, are exactly equal as a consequence of the following equation:

$$F(q) = G(k) \tag{2.18}$$

that can be easily proven using the modular transformations of the functions f_i given in appendix A and eq. (C.32).

This means that the open/closed string duality exactly maps the ultraviolet divergent contribution coming from the massless open string states circulating in the loop — and that reproduces the divergences of $\mathcal{N}=2$ super Yang-Mills living in the world-volume of the fractional D3 branes — into the infrared divergent contribution due to the massless closed string states propagating between the two boundary states. This leads to the first evidence why the one-loop running coupling constant can be consistently derived from a supergravity calculation as originally shown in refs. [21]–[23] and reviewed in ref. [3]. This will be shown in a more direct and quantitative way in the next section.

3. Field theory limit in the two channels

In this section we perform the field theory limit of the amplitudes given by eqs. (2.14) and (2.16) in the open and closed string channel, respectively. In both channels the field theory limit is obtained by performing the zero slope limit ($\alpha' \to 0$) *together* with the limit in which the modular variables t and τ go to infinity, keeping fixed the dimensional Schwinger proper times $\sigma = \alpha'\tau$ and $s = \alpha' t$ of the open and closed string, respectively.

Indeed, in these two limits the only surviving contributions in eqs. (2.14) and (2.16) are those due to the massless states. Notice that the two regions $t \to \infty$ and $\tau \to \infty$ are not connected to each other through a modular transformation.

Let us start performing the field theory limit in the open string channel as explained above, namely by taking $\tau \to \infty$, $\alpha' \to 0$ with $\alpha'\tau \equiv \sigma$ fixed. In so doing we see that in eq. (2.14) the contribution of the massive open string states vanishes and we get:

$$\begin{aligned} Z_h^o(F) \to & \left[-\int d^4x \frac{1}{4} F^a_{\alpha\beta} F^{a\,\alpha\beta} \right] \left[\frac{1}{g^2_{\rm YM}(\Lambda)} - \frac{N}{8\pi^2} \int_{1/\Lambda^2}^{\infty} \frac{d\sigma}{\sigma} e^{-\frac{y^2 \sigma}{2\pi(\alpha')^2}} \right] + \\ & + iN \left[\frac{1}{32\pi^2} \int d^4x F^a_{\alpha\beta} \tilde{F}^{a\,\alpha\beta} \right] \int_{1/\Lambda^2}^{\infty} \frac{d\sigma}{\sigma} e^{-\frac{y^2\sigma}{2\pi(\alpha')^2}} . \end{aligned} \tag{3.1}$$

The previous integrals are naturally regularized in the infrared regime ($\sigma \to \infty$) by the fact that the two stacks of branes are at a finite distance y. Notice that, in order to get a finite expression in the field theory limit, we also need to take the limit $y \to 0$ while keeping fixed the quantity $\frac{y}{\alpha'}$. This finite quantity is directly related to the complex scalar field Ψ of the $\mathcal{N} = 2$ gauge supermultiplet by the gauge/gravity relation [1],

$$\Psi = \frac{y}{2\pi\alpha'} e^{i\theta} \qquad \text{with} \qquad x^4 + ix^5 \equiv y e^{i\theta} , \tag{3.2}$$

and the fact that Ψ has a nonzero vacuum expectation value does not enlarge the gauge group SU(N).

If we perform the field theory limit in the closed string channel by taking $t \to \infty$, $\alpha' \to 0$ with $\alpha' t \equiv s$ fixed, we get from eq. (2.16):

$$\begin{aligned} Z_h^c(F) \to & \left[-\int d^4x \frac{1}{4} F^a_{\alpha\beta} F^{a\,\alpha\beta} \right] \left[\frac{1}{g^2_{\rm YM}(\Lambda)} - \frac{N}{8\pi^2} \int_0^{(\alpha'\Lambda)^2} \frac{ds}{s} e^{-\frac{y^2}{2\pi s}} \right] + \\ & + iN \left[\frac{1}{32\pi^2} \int d^4x F^a_{\alpha\beta} \tilde{F}^{a\,\alpha\beta} \right] \int_0^{(\alpha'\Lambda)^2} \frac{ds}{s} e^{-\frac{y^2}{2\pi s}} . \end{aligned} \tag{3.3}$$

In the closed string case the distance y between the branes makes the integral convergent in the ultraviolet regime ($s \to 0$), but instead an infrared cutoff Λ is needed. If we identify the two Λ's appearing in the ultraviolet cutoff in the open string channel and in the infrared cutoff in the closed string one, we see that the expressions in the two field theory limits are actually equal. This observation clarifies now why the supergravity solution gives the correct answer for the perturbative behavior of the non-conformal world-volume theory as found in refs. [21]–[25]. In fact we can extract the coefficient of the term F^2 from either of the two eqs. (3.1) and (3.3) obtaining the following expression:

$$\frac{1}{g^2_{\rm YM}(\epsilon)} + \frac{N}{8\pi^2} \log \frac{y^2}{\epsilon^2} \equiv \frac{1}{g^2_{\rm YM}(y)} , \qquad \epsilon^2 \equiv 2\pi(\alpha'\Lambda)^2 \tag{3.4}$$

where the integral appearing in eq. (3.1) has been explicitly computed:

$$I(\Lambda, y) \equiv \int_{1/\Lambda^2}^{\infty} \frac{d\sigma}{\sigma} e^{-\frac{y^2\sigma}{2\pi(\alpha')^2}} \simeq \log \frac{2\pi(\alpha'\Lambda)^2}{y^2} . \tag{3.5}$$

Eq. (3.4) exactly reproduces eq. (143) of ref. [3], with $N_2 = 0, N_1 = N$. In that paper the renormalization group parameter μ that describes the running is related to the distance y between the two stacks of branes precisely by the relation $\mu = y/2\pi\alpha'$, while ϵ is an ultraviolet cutoff. The previous derivation makes it clear why the running coupling constant of $\mathcal{N} = 2$ super Yang-Mills can be obtained from the supergravity solution corresponding to a fractional D3 brane of the orbifold $\mathbb{C}^2/\mathbb{Z}_2$.

Eq. (3.4) gives the one-loop correction to the bare gauge coupling constant $g_{\rm YM}(\Lambda)$ in the gauge theory regularized with cutoff Λ. The renormalization procedure can then be performed by introducing the renormalized coupling constant $g_{\rm YM}(\mu)$ related to the bare one by:

$$\frac{1}{g^2_{\rm YM}(\Lambda)} = \frac{1}{g^2_{\rm YM}(\mu)} + \frac{N}{8\pi^2} \log \frac{\Lambda^2}{\mu^2} \tag{3.6}$$

being μ the renormalization scale. Using eq. (3.6) in either eq. (3.1) or eq. (3.3) one gets the following expression for the coefficient of the F^2 term

$$\frac{1}{g^2_{\rm YM}(\mu)} - \frac{N}{8\pi^2} \log \frac{\mu^2}{m^2} = \frac{1}{g^2_{\rm YM}(m)}\,; \qquad m^2 \equiv \frac{y^2}{2\pi\alpha'^2}\,. \tag{3.7}$$

From it, or equivalently from eq. (3.6), we can now determine the one-loop β-function:

$$\beta(g_{\rm YM}) \equiv \mu \frac{\partial}{\partial\mu} g_{\rm YM}(\mu) = -\frac{g^3_{\rm YM} N}{8\pi^2} \tag{3.8}$$

which is the correct one for $\mathcal{N} = 2$ super Yang-Mills.

Let us turn now to the vacuum angle $\theta_{\rm YM}$, provided by the terms in eqs. (3.1) and (3.3) with the topological charge. If we extract it from either of these two equations we find that it is imaginary and, moreover, must be renormalized like the coupling constant. A way of eliminating these problems is to introduce a complex cutoff and to allow the gauge field to be in either one of the two sets of branes by taking the symmetric combination:[4]

$$\frac{1}{2}\left[\langle D3;F|D|D3\rangle + \langle D3|D|D3;F\rangle\right] = \frac{1}{2}\left[\langle D3;F|D|D3\rangle + \overline{\langle D3;F|D|D3\rangle}\right]. \tag{3.9}$$

If we introduce a complex cutoff $\Lambda \to \Lambda e^{-i\theta}$ the divergent integral in eq. (3.5) becomes:

$$I(z) \equiv \int_{1/\Lambda^2}^{\infty} \frac{d\sigma}{\sigma} e^{-\frac{y^2\sigma}{2\pi(\alpha')^2}} \simeq \log \frac{2\pi(\alpha')^2\Lambda^2}{y^2 e^{2i\theta}}\,. \tag{3.10}$$

This procedure leaves unchanged all the previous considerations concerning the gauge coupling constant, because in this case the coefficient of the F^2 term results to be proportional to the following combination:

$$\frac{1}{2}\left[I(z) + I(\bar{z})\right] \simeq \log \frac{2\pi(\alpha')^2\Lambda^2}{y^2}\,. \tag{3.11}$$

In the case of the $\theta_{\rm YM}$ angle one gets instead:

$$\theta_{\rm YM} = i\frac{N}{2}\left[I(z) - I(\bar{z})\right] = 2N\theta\,, \tag{3.12}$$

[4]We thank M. Billó, F. Lonegro and I. Pesando for useful discussions on this point.

exactly reproducing the result given in eq. (144) of ref. [3] if we take again $N_2 = 0$ and $N_1 = N$. Remember, however, that in ref. [3] θ is the phase of the complex quantity $z = ye^{i\theta}$. But, as it can be seen in eq. (3.10), giving a phase to the cutoff corresponds to giving the opposite phase to the distance between the branes y. We prefer to make the cutoff complex rather than y in order to keep the open string Virasoro generator L_0 real.

4. Branes in external fields: $\mathcal{N} = 1$ orbifold

In the following we extend the analysis performed in the previous section to the case of the orbifold $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ preserving four supersymmetry charges. The orbifold group $\mathbb{Z}_2 \times \mathbb{Z}_2$ contains four elements whose action on the three complex coordinates

$$z_1 = x_4 + ix_5 \qquad z_2 = x_6 + ix_7 \qquad z_3 = x_8 + ix_9 \tag{4.1}$$

is chosen to be

$$\begin{aligned} R_e &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad R_{h_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \\ R_{h_2} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad R_{h_3} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{4.2}$$

As previously stated, fractional branes have Chan-Paton factors transforming according to irreducible representations of the orbifold group. The group $\mathbb{Z}_2 \times \mathbb{Z}_2$ has four irreducible one-dimensional representations corresponding to four different kinds of fractional branes. The orbifold $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, as already explained in refs. [13, 29], can be seen as obtained by three copies of the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ where the i-th $\mathbb{Z}_2$ contains the elements (e, h_i) $(i = 1, \ldots 3)$. This means that the boundary states associated to each fractional brane are:

$$\begin{aligned} |Dp\rangle_1 &= |Dp\rangle_u + |Dp\rangle_{t_1} + |Dp\rangle_{t_2} + |Dp\rangle_{t_3}\,, \\ |Dp\rangle_2 &= |Dp\rangle_u + |Dp\rangle_{t_1} - |Dp\rangle_{t_2} - |Dp\rangle_{t_3}\,, \\ |Dp\rangle_3 &= |Dp\rangle_u - |Dp\rangle_{t_1} + |Dp\rangle_{t_2} - |Dp\rangle_{t_3}\,, \\ |Dp\rangle_4 &= |Dp\rangle_u - |Dp\rangle_{t_1} - |Dp\rangle_{t_2} + |Dp\rangle_{t_3}\,, \end{aligned} \tag{4.3}$$

where $|Dp\rangle_u$ is the untwisted boundary state that, apart an overall factor 1/2 due to the orbifold projection, is the same as the one in flat space, and $|Dp\rangle_{t_i}$ $(i = 1, \ldots, 3)$ are exactly the same as the twisted boundary states on the orbifold $\mathbb{C}^2/\mathbb{Z}_2$, apart from a factor $1/\sqrt{2}$. The signs in front of each twisted term in eq. (4.3) depend on the irreducible representation chosen for the orbifold group action on the Chan-Paton factors.

In order to keep the forthcoming discussion as general as possible, we study the interaction between a stack of N_I $(I = 1, \ldots, 4)$ branes of type I and a D3-fractional brane, for example of type $I = 1$, with an SU(N) gauge field turned on its world-volume. Due to the structure of the orbifold $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, this interaction is the sum of four terms:

$$Z = Z_e + \sum_{i=1}^{3} Z_{h_i} \,, \tag{4.4}$$

where Z_e and Z_{h_i} are obtained in the open [closed] channel by multiplying the eqs. (2.4) and (2.5) [eqs. (2.9) and (2.10)] by an extra 1/2 factor due to the orbifold projection. Also in this case we limit our considerations to the twisted sectors, which are the only ones that provide a non-zero contribution to the quadratic terms in the gauge field. Their contribution, in the open string channel, is given by:

$$\begin{aligned} Z^o_{h_i} = & \frac{f_i(N)}{2\,(8\pi^2\alpha')^2} \int d^4x \sqrt{-\det(\eta + \hat{F})} \int_0^\infty \frac{d\tau}{\tau} e^{-\frac{y_i^2\tau}{2\pi\alpha'}} \frac{4\,\sin\pi\nu_f \sin\pi\nu_g}{\Theta_2^2(0|i\tau)\Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)} \times \\ & \times \left\{\Theta_3^2(0|i\tau)\Theta_4(i\nu_f\tau|i\tau)\Theta_4(i\nu_g\tau|i\tau) - \Theta_4^2(0|i\tau)\Theta_3(i\nu_f\tau|i\tau)\Theta_3(i\nu_g\tau|i\tau)\right\} + \\ & + \frac{i\,f_i(N)}{64\pi^2} \int d^4x F^a_{\alpha\beta} \tilde{F}^{a\,\alpha\beta} \int \frac{d\tau}{\tau} e^{-\frac{y_i^2\tau}{2\pi\alpha'}} \,. \end{aligned} \tag{4.5}$$

In the previous expression we should put to zero the distance y_i between the stack of the N_I's branes and the dressed one, since the fractional branes are constrained to live at the orbifold fixed point $z_1 = z_2 = z_3 = 0$. However, y_i provides a natural infrared cutoff in eq. (4.5). Therefore we keep this quantity small but finite, and we will put it to zero just at the end of the calculation. The functions $f_i(N)$ introduced in eq. (4.5) depend on the number of the different kinds of fractional branes N_I, and their explicit expression is given by [13, 29]:

$$\begin{aligned} f_1(N_I) &= N_1 + N_2 - N_3 - N_4 \,, \\ f_2(N_I) &= N_1 - N_2 + N_3 - N_4 \,, \\ f_3(N_I) &= N_1 - N_2 - N_3 + N_4 \,. \end{aligned} \tag{4.6}$$

One can follow the same steps in the closed string channel obtaining the formulas corresponding to eq. (2.10).

Let us now extract in both channels the quadratic terms in the gauge field F. In the open sector, we get:

$$\begin{aligned} Z^o_h(F) \to & \left[-\frac{1}{4}\int d^4x F^a_{\alpha\beta} F^{a\,\alpha\beta}\right] \times \\ & \times \left\{\frac{1}{g^2_{\mathrm{YM}}(\Lambda)} - \sum_{i=1}^{3} \frac{f_i(N)}{16\pi^2}\left[\int_{1/(\alpha'\Lambda^2)}^\infty \frac{d\tau}{\tau} e^{-\frac{y_i^2\tau}{2\pi\alpha'}} + \int_0^\infty \frac{d\tau}{\tau} e^{-\frac{y_i^2\tau}{2\pi\alpha'}} G(k)\right]\right\} + \\ & + i\left[\frac{1}{32\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\right] \sum_{i=1}^{3} \frac{f_i(N)}{2} \int_{\frac{1}{\alpha'\Lambda^2}}^\infty \frac{d\tau}{\tau} e^{-\frac{y_i^2\tau}{2\pi\alpha'}} \,, \end{aligned} \tag{4.7}$$

while in the closed string channel we obtain:

$$Z^c_h(F) \to \left[-\frac{1}{4}\int d^4x F^a_{\alpha\beta} F^{a\,\alpha\beta}\right] \times$$

$$\times \left\{ \frac{1}{g^2_{\rm YM}(\Lambda)} - \sum_{i=1}^{3} \frac{f_i(N)}{16\pi^2} \left[\int_0^{\alpha'\Lambda^2} \frac{dt}{t} e^{-\frac{y_i^2}{2\pi\alpha' t}} + \int_0^{\infty} \frac{dt}{t} e^{-\frac{y_i^2}{2\pi\alpha' t}} F(q) \right] \right\} +$$
$$+ i \left[\frac{1}{32\pi^2} \int d^4x F^a_{\alpha\beta} \tilde{F}^{a\,\alpha\beta} \right] \sum_{i=1}^{3} \frac{f_i(N)}{2} \int_0^{\alpha'\Lambda^2} \frac{dt}{t} e^{-\frac{y_i^2}{2\pi\alpha' t}} \,. \tag{4.8}$$

Analogously to the case of the $\mathcal{N} = 2$ orbifold, we have isolated, in both channels, the divergent contribution due to the massless states, and we have also added the one coming from the tree diagrams. The main properties exhibited by the interactions in the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ in eqs. (2.14) and (2.16) are also shared by the interactions in the orbifold $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ in eqs. (4.7) and (4.8). In particular, also in this case, one can see that the contribution of the massless states and that of the massive states transform respectively into each other without any mixing between them. This confirms the main result obtained in section 2, i.e. the open/closed string duality exactly maps the ultraviolet divergent contribution coming from the massless open string states — which in this case reproduces the divergences of $\mathcal{N} = 1$ super Yang-Mills — into the infrared divergent contribution due to the massless closed string states.

Extracting the coefficient of the term F^2 from eq. (4.7) or (4.8), and performing on it the field theory limit as explained in the previous section, we get:

$$\frac{1}{g^2_{\rm YM}(\epsilon)} + \frac{1}{8\pi^2} \sum_{i=1}^{3} f_i(N_I) \log \frac{y_i}{\epsilon} \equiv \frac{1}{g^2_{\rm YM}}(y) \qquad \epsilon^2 = 2\pi(\alpha'\Lambda)^2 \,. \tag{4.9}$$

Eq. (4.9) reproduces eq. (3.14) of ref. [8], and this explains again why the supergravity solutions, dual to $\mathcal{N} = 1$ super Yang-Mills theory, give the correct answer for the perturbative behavior of the non conformal world-volume theory, as found in refs. [8, 13, 29].

Performing the renormalization procedure, we introduce the renormalized coupling constant $g_{\rm YM}(\mu)$ given in terms of the bare one by the relation:

$$\frac{1}{g^2_{\rm YM}(\Lambda)} = \frac{1}{g^2_{\rm YM}(\mu)} + \sum_{i=1}^{3} \frac{f_i(N)}{16\pi^2} \log \frac{\Lambda^2}{\mu^2} = \frac{1}{g^2_{\rm YM}(\mu)} + \frac{3N_1 - N_2 - N_3 - N_4}{16\pi^2} \log \frac{\Lambda^2}{\mu^2} \,. \tag{4.10}$$

From this equation we can determine the β-function obtaining:

$$\beta(g_{\rm YM}) \equiv \mu \frac{\partial}{\partial\mu} g_{\rm YM}(\mu) = - \frac{g^3_{\rm YM}}{16\pi^2} \left(3N_1 - N_2 - N_3 - N_4\right) , \tag{4.11}$$

that is the correct one for the wold-volume theory living on the dressed brane.

Finally, in the same spirit as in section 3, we consider the symmetric combination given in eq. (3.9) and, by introducing a complex cut-off $\Lambda e^{-i\theta}$, or equivalently, considering complex coordinates $z_i = y_i e^{i\theta}$, we arrive to the following expression for the $\theta_{\rm YM}$:

$$\theta_{\rm YM} = \sum_{i=1}^{3} f_i(N_I)\theta \tag{4.12}$$

again in agreement with the result given in eq. (3.14) of the ref. [8].

5. Conclusions

In this paper we have used open/closed string duality of the annulus diagram for explaining why the perturbative properties of $\mathcal{N} = 1, 2$ Super Yang-Mills theories, living in the world-volume of fractional D3 branes, follow directly from their classical supergravity solutions [21]–[25]. This is a consequence of the fact that the coefficient of the gauge kinetic term in the annulus diagram is expressed by an integral that is divergent already at the string level and, therefore, must be regularized. It turns out that this divergent contribution can be seen to be equivalently due either to the exchange of massless closed strings between two boundary states in the closed string channel or to the massless open string states circulating in the loop in the open string channel, and that these two contributions are mapped into each other by the modular transformation that relates the open and closed string channels. This means that the divergence present at the string level is precisely the one that one gets in the gauge field theory living on the brane in the field theory limit ($\alpha' \to 0$). It remains unclear why these two kinds of divergences must be related. This is in fact not the case, for instance, of the tadpoles present in the bosonic string. Here in fact we have a dilaton tadpole that is not related to the divergences due to the massless gauge fields circulating in the loop in pure Yang-Mills theory for $d = 26$, which are proportional to $d - 26$ [9, 20] and therefore vanish for $d = 26$. Supersymmetry may play a role in the identification of the two divergences [27]. It must also be noticed that the situation here is different from what was found for the gauge kinetic term in the case of the heterotic string [30], where no ultraviolet, but only infrared divergences are found at the string level and these divergences are due to the contribution of the massless string states that are also present in the limiting field theory.

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A. Θ functions

The Θ-functions which are the solutions of the heat equation

$$\frac{\partial}{\partial t}\Theta\left(\nu|it\right) = \frac{1}{4\pi}\partial_\nu^2\Theta(\nu|it) \tag{A.1}$$

are given by:

$$\Theta_1(\nu|it) \equiv \Theta_{11}(\nu,|it) = -2q^{\frac{1}{4}}\sin\pi\nu\prod_{n=1}^{\infty}\left[(1-q^{2n})(1-e^{2i\pi\nu}q^{2n})(1-e^{-2i\pi\nu}q^{2n})\right],$$

$$\Theta_2(\nu|it) \equiv \Theta_{10}(\nu,|it) = 2q^{\frac{1}{4}}\cos\pi\nu\prod_{n=1}^{\infty}\left[(1-q^{2n})(1+e^{2i\pi\nu}q^{2n})(1+e^{-2i\pi\nu}q^{2n})\right],$$

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$$\Theta_3(\nu,|it) \equiv \Theta_{00}(\nu,|it) = \prod_{n=1}^{\infty}\left[(1-q^{2n})(1+e^{2i\pi\nu}q^{2n-1})(1+e^{-2i\pi\nu}q^{2n-1})\right],$$
$$\Theta_4(\nu,|it) \equiv \Theta_{01}(\nu,|it) = \prod_{n=1}^{\infty}\left[(1-q^{2n})(1-e^{2i\pi\nu}q^{2n-1})(1-e^{-2i\pi\nu}q^{2n-1})\right], \tag{A.2}$$

with $q = e^{-\pi t}$. The modular transformation properties of the Θ functions are

$$\Theta_1(\nu|it) = i\,\Theta_1\left(-i\frac{\nu}{t}|\frac{i}{t}\right) e^{-\pi\nu^2/t}t^{-\frac{1}{2}},$$
$$\Theta_{2,\,3,\,4}(\nu|it) = \Theta_{4,\,3,\,2}\left(-i\frac{\nu}{t}|\frac{i}{t}\right) e^{-\pi\nu^2/t}t^{-\frac{1}{2}}. \tag{A.3}$$

It is also useful to define the f-functions and their transformation properties. We define

$$f_1 \equiv q^{\frac{1}{12}}\prod_{n=1}^{\infty}(1-q^{2n}), \tag{A.4}$$
$$f_2 \equiv \sqrt{2}q^{\frac{1}{12}}\prod_{n=1}^{\infty}(1+q^{2n}), \tag{A.5}$$
$$f_3 \equiv q^{-\frac{1}{24}}\prod_{n=1}^{\infty}(1+q^{2n-1}), \tag{A.6}$$
$$f_4 \equiv q^{-\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^{2n-1}). \tag{A.7}$$

In the case of a real argument $q = e^{-\pi t}$ they transform as follows under the modular transformation $t \to 1/t$:

$$f_1(e^{-\frac{\pi}{t}}) = \sqrt{t}f_1(e^{-\pi t}), \qquad f_2(e^{-\frac{\pi}{t}}) = f_4(e^{-\pi t}), \qquad f_3(e^{-\pi t}) = f_3(e^{-\frac{\pi}{t}}). \tag{A.8}$$

The following relations are also useful:

$$\Theta_{2,3,4}(0|it) = f_1(e^{-\pi t})\,f_{2,3,4}^2(e^{-\pi t}); \qquad \lim_{\nu\to 0}\frac{\Theta_1(\nu|it)}{2\sin\pi\nu} = f_1^3(e^{-\pi t}). \tag{A.9}$$

B. Open string channel

In this appendix we compute the one-loop vacuum amplitude in the open string channel, by extending the techniques developed in ref. [31]. We start by considering the action of an open string in a background represented by an $\mathrm{SU}(N)$ gauge field A_μ^a, being μ and a respectively the Lorentz and gauge indices. We choose only one end of the string to be charged under the gauge field, say the one parameterized with $\sigma = 0$, and, for the sake of simplicity, we consider a gauge field pointing along a definite direction in the gauge group, so that we can write $A_\nu^a = \frac{1}{2}F_{\mu\nu}^a X^\mu$ where $F_{\mu\nu}$ is taken constant. The string action turns out to be:

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\left[\eta^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu - i\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu\right] + \frac{1}{2}\int d\tau F_{\mu\nu}\left[X^\nu\partial_\tau X^\mu - \frac{i}{2}\bar{\psi}^\nu\rho^0\psi^\mu\right]. \tag{B.1}$$

Since the external field couples to the boundary, its effect is to modify the boundary conditions of the string coordinates. In order to determine such modifications it is convenient to introduce the following sets of coordinates:

$$X_f^{\pm} \equiv \frac{X^0 \pm X^1}{\sqrt{2}} \quad \text{and} \quad X_g^{\pm} \equiv \frac{X^2 \pm i X^3}{\sqrt{2}}\,; \tag{B.2}$$

$$\psi_{f\;R,L}^{\pm} \equiv \frac{\psi^0_{R,L} \pm \psi^1_{R,L}}{\sqrt{2}} \quad \text{and} \quad \psi_{g\;R,L}^{\pm} \equiv \frac{\psi^2_{R,L} \pm i\psi^3_{R,L}}{\sqrt{2}}\,, \tag{B.3}$$

where ψ_R and ψ_L are the two components with opposite chirality of the Majorana-Weyl world-sheet spinor ψ.

The surface terms, that arise in the variation of the action in eq. (B.1) with a field strength F given by eq. (2.2), vanish if the string coordinates, previously introduced, satisfy the conditions:

$$\partial_\sigma X_f^{\pm}|_{\sigma=0} = \mp f \partial_\tau X_f^{\pm}|_{\sigma=0}\,, \qquad \partial_\sigma X_g^{\pm}|_{\sigma=0} = \mp i g \partial_\tau X_g^{\pm}|_{\sigma=0}\,, \tag{B.4}$$

$$\psi_{f\;R}^{\pm}|_{\sigma=0} = \frac{(1 \pm f)}{(1 \mp f)} \psi_{f\;L}^{\pm}|_{\sigma=0}\,, \qquad \psi_{g\;R}^{\pm}|_{\sigma=0} = \frac{(1 \pm ig)}{(1 \mp ig)} \psi_{g\;L}^{\pm}|_{\sigma=0} \tag{B.5}$$

at $\sigma = 0$, together with the standard boundary conditions at $\sigma = \pi$

$$\partial_\sigma X_f^{\pm}|_{\sigma=\pi} = 0\,, \qquad \partial_\sigma X_g^{\pm}|_{\sigma=\pi} = 0\,, \tag{B.6}$$

$$\psi_{f\;R}^{\pm}|_{\sigma=\pi} + (-1)^a \psi_{f\;L}^{\pm}|_{\sigma=\pi} = 0\,, \qquad \psi_{g\;R}^{\pm}|_{\sigma=\pi} + (-1)^a \psi_{g\;L}^{\pm}|_{\sigma=\pi} = 0\,, \tag{B.7}$$

with $a = 0, 1$ in the NS and R sector, respectively. The mode expansions of the bosonic coordinates in eq. (B.2) take the form

$$X_f^{\pm}(\sigma,\tau) = x_f^{\pm} + i\sqrt{2\alpha'} \sum_{n \in Z} \frac{\alpha^{\pm}_{f\,n}}{(n \pm i\epsilon_f)} \phi^{\pm}_{f\,n}(\sigma,\tau)\,, \tag{B.8}$$

where the oscillators modes are defined as follows:

$$\alpha^{\pm}_{f\,n} = \sqrt{n \pm i\epsilon_f}\, a_n^{\pm} \quad n \geq 0 \qquad \text{and} \qquad \alpha^{\pm}_{f\,-n} = \sqrt{n \mp i\epsilon_f}\, a_{-n}^{\pm} \qquad n > 0 \tag{B.9}$$

$$\phi^{\pm}_{f\,n} = e^{-i(n \pm i\epsilon_f)\tau} \cos\left[(n \pm i\epsilon_f)\sigma \mp i\pi\epsilon_f\right]. \tag{B.10}$$

For the coordinate X_g we simply replace the index f with g and the Fourier modes a_n with b_n. In the previous equations we have defined:

$$\epsilon_f \equiv \frac{1}{\pi}\mathrm{arcth} f\,, \qquad \epsilon_g \equiv \frac{1}{\pi}\mathrm{arcth}(ig)\,. \tag{B.11}$$

Analogously, the mode expansions of the fermionic coordinates, satisfying the conditions in eqs. (B.5) and (B.7), turn out to be:

$$\psi_{f\;R,L}^{\pm} = \sqrt{2\alpha'} \sum_{n \in \mathcal{Z} + \frac{a+1}{2}} d_n^{\pm} \chi^{\pm}_{f\,n\;R,L}(\sigma,\tau) \qquad \psi_{g\;R,L}^{\pm} = \sqrt{2\alpha'} \sum_{n \in \mathcal{Z} + \frac{a+1}{2}} h_n^{\pm} \chi^{\pm}_{g\,n\;R,L}(\sigma,\tau) \tag{B.12}$$

with

$$\chi^{\pm}_{f\,n\,R}(\sigma,\tau) = \frac{1}{\sqrt{2}} e^{-i(n\pm i\epsilon_f)(\tau-\sigma)\pm\pi\epsilon_f}\,, \qquad \chi^{\pm}_{f\,n\,L}(\sigma,\tau) = \frac{1}{\sqrt{2}} e^{-i(n\pm i\epsilon_f)(\tau+\sigma)\mp\pi\epsilon_f}\,. \tag{B.13}$$

The Fourier modes for the fields $\psi^{\pm}_g$ are, again, obtained by replacing the index f with g. Furthermore, it is useful in the forthcoming discussion, to give also the relations between the parameters defined in eq. (2.6) with the one given in eq. (B.11):

$$\epsilon_f = i\nu_f \qquad \text{and} \qquad \epsilon_g = -i\nu_g\,. \tag{B.14}$$

The canonical quantization procedure leads to the following commutation relations for the Fourier modes:

$$[x^+_f, x^-_f] = -i\frac{2\alpha'\pi}{f} \qquad [a^+_n, a^-_{-n}] = -1 \qquad \{d^+_n, d^-_m\} = -\delta_{n+m} \tag{B.15}$$

$$[x^+_g, x^-_g] = \frac{2\alpha'\pi}{g} \qquad [b^+_n, b^-_{-n}] = 1 \qquad \{h^+_n, h^-_n\} = \delta_{n+m}\,. \tag{B.16}$$

Furthermore, with the help of eqs. (B.8) and (B.12), one can compute the Virasoro generator L_0:

$$\begin{aligned} L_0 = L^{\perp}_0 &- \sum_{n=-\infty}^{+\infty} :\alpha^{+}_{f\,n}\cdot\alpha^{-}_{f\,-n}: + \sum_{n=-\infty}^{+\infty} :\alpha^{+}_{g\,n}\cdot\alpha^{-}_{g\,-n}: + c(a) - \\ &- \sum_{n\in\mathbf{Z}+\frac{\mathbf{a+1}}{\mathbf{2}}} (n+i\epsilon_f) : d^-_{-n}\cdot d^+_n : + \sum_{n\in\mathbf{Z}+\frac{\mathbf{a+1}}{\mathbf{2}}} (n+i\epsilon_g) : h^-_{-n}\cdot h^+_n : \end{aligned} \tag{B.17}$$

where $L^{\perp}_0$ denotes the standard contribution coming from the direction orthogonal to the brane, and the constant $c(a)$ is the zero point energy, whose value is corrected by the presence of the gauge field [31]:

$$c(0) = \frac{1}{2}i\epsilon_f(1-i\epsilon_f) + \frac{1}{2}i\epsilon_g(1-i\epsilon_g) - \frac{\epsilon_f^2}{2} - \frac{\epsilon_g^2}{2} - \frac{1}{2}\,, \tag{B.18}$$

$$c(1) = 0\,. \tag{B.19}$$

Now we have all the ingredients to compute the one-loop free energy given in eq. (2.3). It is the sum of six terms:

$$Z = Z^{NS}_e + Z^{NS(-1)^F}_e + Z^R_e + Z^{NS}_g + Z^{NS(-1)^F}_g + Z^{R(-1)^f}_g\,. \tag{B.20}$$

In order to evaluate each term, we notice that the external field causes essentially two modifications in the one-loop vacuum amplitude in eq. (2.3) with respect to the case without gauge field. One concerns the oscillation frequencies of the longitudinal coordinates and the other the zero mode contributions. In particular it is easy to see that the oscillation frequencies get shifted by $\pm i\epsilon_f$ in the $0,1$ plane and $\pm i\epsilon_g$ in the $(2,3)$ plane:

$$\prod_{n=1}^{\infty}\left(\frac{1}{1-k^{2n}}\right)^8 \longrightarrow \prod_{n=1}^{\infty}\left(\frac{1}{1-k^{2n}}\right)^4\left(\frac{1}{1-k^{2n}e^{2\pi i\tau\epsilon_f}}\right)\left(\frac{1}{1-k^{2n}e^{-2\pi i\tau\epsilon_f}}\right)\times \\ \times\left(\frac{1}{1-k^{2n}e^{2\pi i\tau\epsilon_g}}\right)\left(\frac{1}{1-k^{2n}e^{-2\pi i\tau\epsilon_g}}\right) \tag{B.21}$$

with $k=e^{-\pi\tau}$. Analogous modifications occur in the fermionic calculation. The contribution of the bosonic zero modes to the partition function, instead, requires some care because of the anomalous commutation relations satisfied by the coordinates $x^{\pm}_{(f;g)}$, explicitly given in eq. (B.15). Due to them, as explained in ref. [31], we have to compute the density of the quantum states. By analogy with the case of the conjugate variables $[x,p]=i\frac{h}{2\pi}$, where such a density is simply given by $\rho=1/h$, from eqs. (B.15) and (B.16), we deduce in our case the following expression:

$$\rho=-i\frac{f\,g}{(4\pi^2\alpha')^2}\,. \tag{B.22}$$

The contribution of the bosonic zero modes to the free energy is given by:

$$\mathrm{Tr}\left[e^{-2\pi\tau\left(\frac{y^2}{(2\pi)^2\alpha'}-i\epsilon_f a_0^- a_0^+ + i\epsilon_g b_0^- b_0^+\right)}\right]=e^{-\frac{y^2\tau}{2\pi\alpha'}}\left[\frac{1}{1-e^{-2\pi i\tau\epsilon_f}}\right]\left[\frac{1}{1-e^{-2\pi i\tau\epsilon_g}}\right].$$

Let us consider the fermionic zero mode contribution to the free energy arising from the Ramond sector. It is well known that it is divergent and must be regularized [32]. It is more convenient to perform the regularization in the euclidean space. At this aim we first introduce the following operators:

$$\psi_0^0=\frac{d_0^+ + d_0^-}{\sqrt{2}}\,,\qquad \psi_0^1=\frac{d_0^+ - d_0^-}{\sqrt{2}}\,,\qquad \psi_0^2=\frac{h_0^+ + h_0^-}{\sqrt{2}}\,,\qquad \psi_0^3=\frac{h_0^+ - h_0^-}{\sqrt{2}i} \tag{B.24}$$

that, together with the zero modes associated to the transverse directions ψ_0^i $(j=4,\dots 9)$, satisfy the usual anticommutation rules $\{\psi_0^\mu\psi_0^\nu\}=\eta^{\mu\nu}$. Then we perform a Wick rotation $\psi_0^0=i\psi_0^{10}$. Furthermore, it is convenient to introduce the raising and lowering operators defined as follows:

$$e_1^{\pm}=\pm d_0^{\pm}\,;\qquad e_2^{\pm}=h_0^{\pm}\qquad e_j^{\pm}=\frac{\psi_0^{2j-2}\pm i\psi_0^{2j-1}}{\sqrt{2}}\quad \text{with}\quad j=3\dots 5 \tag{B.25}$$

satisfying the algebra:

$$\{e_{\mathbf{a}}^+,e_{\mathbf{b}}^-\}=\delta_{\mathbf{a},\mathbf{b}}\,. \tag{B.26}$$

The Hilbert space, associated to each couple of operators in eq. (B.25), is two-dimensional and it is spanned by the states $|s_{\mathbf{a}}=\pm 1\rangle$, being s the eigenvalues of the number operators

$$N_{\mathbf{a}}=[e_{\mathbf{a}}^+,e_{\mathbf{a}}^-]\,. \tag{B.27}$$

In terms of these latter, the fermionic zero modes in the R-sector become

$$d_0^- d_0^+=\frac{N_1-1}{2}\,,\qquad h_0^- h_0^+=\frac{1-N_2}{2} \tag{B.28}$$

and the GSO, projector together with the orbifold action, takes the following form [3, 33]:

$$(-1)^{F_0} = \prod_{k=1}^{5} N_k\,, \qquad g = e^{i\frac{\pi}{2}(N_4+N_5)} = -N_4 N_5\,. \tag{B.29}$$

Let us compute explicitly the traces over the fermionic zero modes appearing in eq. (2.3). In particular we have:

$$\mathrm{Tr}_R\left(e^{-2\pi\tau(-i\epsilon_f d_0^- d_0^+ + i\epsilon_g h_0^- h_0^+)}(-1)^{G_{\beta\gamma}}\right) = 4\left(1+e^{-i\pi\tau\epsilon_f}\right)\left(1+e^{-i\pi\tau\epsilon_g}\right) \tag{B.30}$$

and

$$\begin{aligned}\mathrm{Tr}_R&\left(e^{-2\pi\tau(-i\epsilon_f d_0^- d_0^+ + i\epsilon_g h_0^- h_0^+)} g(-1)^{F_0}\right) = \\ &= -\lim_{x\to 1}\mathrm{Tr}_R\left(e^{i\pi\tau[\epsilon_f(N_1-1)-\epsilon_g(1-N_2)]}\prod_{k=1}^{5} x^{N_k} N_1 N_2 N_3\right)\mathrm{Tr}\left(x^{-2\gamma_0\beta_0}\right)\end{aligned} \tag{B.31}$$

where we have introduced the regulator $x^{\sum_k N_k} x^{-2\gamma_0\beta_0}$ in order to have a finite result. Therefore by using the following relations:

$$\mathrm{Tr}\left(x^{N_k}N_k\right) = \left(x - \frac{1}{x}\right), \qquad \mathrm{Tr}\left(x^{N_k}\right) = \left(x + \frac{1}{x}\right) \tag{B.32}$$

$$\mathrm{Tr}\left(e^{i\pi\tau\epsilon_f N_k} x^{N_k} N_k\right) = \left(x e^{i\pi\tau\epsilon_f} - \frac{1}{x e^{i\pi\tau\epsilon_f}}\right), \qquad \mathrm{Tr}\left(x^{-2\gamma_0\beta_0}\right) = \frac{1}{1-x^2}\,, \tag{B.33}$$

we get that the zero mode contribution of the $R(-1)^F$ sector to Z_g is

$$\mathrm{Tr}_R\left(e^{-2\pi\tau(-i\epsilon_f d_0^- d_0^+ + i\epsilon_g h_0^- h_0^+)} g(-1)^{F_0}\right) = -16 e^{-\pi i\tau\epsilon_f} e^{-i\pi\tau\epsilon_g}\sin\pi\tau\epsilon_f \sin\pi\tau\epsilon_g. \tag{B.34}$$

Finally, by collecting all the results, and observing that in eq. (B.22) one can write

$$if\,g = \sin\pi\nu_f \sin\pi\nu_g \sqrt{-\det(\eta + \hat{F})} \tag{B.35}$$

we get the following expressions for the various terms defined in eq. (B.20):

$$Z = -N\int d^4x \sqrt{-\det\left(\eta+\hat{F}\right)}\left[\frac{\sin\pi\nu_f \sin\pi\nu_g}{(4\pi^2\alpha')^2}\right]\tilde{Z}$$

with

$$\tilde{Z}_e^{NS} = \frac{1}{4}\int_0^\infty \frac{d\tau}{\tau} e^{-\frac{y^2\tau}{2\pi\alpha'}} \frac{f_3^4(k)\Theta_3(i\nu_f\tau|i\tau)\Theta_3(i\nu_g\tau|i\tau)}{f_1^4(k)\Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)}\,, \tag{B.37}$$

$$\tilde{Z}_e^{NS(-1)^F} = -\frac{1}{4}\int_0^\infty \frac{d\tau}{\tau} e^{-\frac{y^2\tau}{2\pi\alpha'}} \frac{f_4^4(k)\Theta_4(i\nu_f\tau|i\tau)\Theta_4(i\nu_g\tau|i\tau)}{f_1^4(k)\Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)}\,, \tag{B.38}$$

$$\tilde{Z}_e^{R} = -\frac{1}{4}\int_0^\infty \frac{d\tau}{\tau} e^{-\frac{y^2\tau}{2\pi\alpha'}} \frac{f_2^4(k)\Theta_2(i\nu_f\tau|i\tau)\Theta_2(i\nu_g\tau|i\tau)}{f_1^4(k)\Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)}\,, \tag{B.39}$$

$$\tilde{Z}_h^{NS} = \int_0^\infty \frac{d\tau}{\tau} e^{-\frac{y^2\tau}{2\pi\alpha'}} \frac{\Theta_4^2(0|i\tau)\Theta_3(i\nu_f\tau|i\tau)\Theta_3(i\nu_g\tau|i\tau)}{\Theta_2^2(0|i\tau)\Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)}\,, \tag{B.40}$$

$$\tilde{Z}_h^{NS(-1)^F} = -\int_0^\infty \frac{d\tau}{\tau} e^{-\frac{y^2\tau}{2\pi\alpha'}} \frac{\Theta_3^2(0|i\tau)\Theta_4(i\nu_f\tau|i\tau)\Theta_4(i\nu_g\tau|i\tau)}{\Theta_2^2(0|i\tau)\Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)}, \tag{B.41}$$

while

$$Z_h^{R(-1)^F} = iN \int d^4x \frac{fg}{(4\pi^2\alpha')^2} \int_0^\infty \frac{d\tau}{\tau} e^{-\frac{y^2\tau}{2\pi\alpha'}}. \tag{B.42}$$

The last contribution can be written in terms of $F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}$, with $\tilde{F}^a_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F^{a\,\gamma\delta}$, using the relation

$$f\,g = \frac{(2\pi\alpha')^2}{8} F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}, \tag{B.43}$$

giving the last term in eq. (2.5).

In the final part of this appendix we expand eq. (2.5) up to quadratic terms in the gauge field. We need the following expansions:

$$\Theta_3(i\nu_f\tau|i\tau) = \Theta_3(0|i\tau)\left[1 - 4f^2\tau^2\sum_{n=1}^\infty \frac{k^{2n-1}}{(1+k^{2n-1})^2} + \ldots\right], \tag{B.44}$$

$$\Theta_4(i\nu_f\tau|i\tau) = \Theta_4(0|i\tau)\left[1 + 4f^2\tau^2\sum_{n=1}^\infty \frac{k^{2n-1}}{(1-k^{2n-1})^2} + \ldots\right] \tag{B.45}$$

and

$$\lim_{\nu\to 0}\frac{2\sin\pi\nu}{\Theta_1(i\nu\tau|i\tau)} = -\frac{1}{i\tau f_1^3(k)}. \tag{B.46}$$

Inserting them in eq. (2.5), reintroducing the field strength of the gauge field by means of the following equation:[5]

$$f^2 - g^2 = -\frac{(2\pi\alpha')^2}{4} F^a_{\alpha\beta}F^{a\alpha\beta} \tag{B.47}$$

and using eq. (B.43), we get for the term quadratic in the gauge field the following expression:

$$\begin{aligned} Z_h^o\left(F^2\right) = &-\frac{N}{8\pi^2}\int d^4x\left[-\frac{1}{4}F^a_{\alpha\beta}F^{a\alpha\beta}\right]\int_0^\infty \frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}} \times \\ &\times \frac{1}{2}\prod_{n=1}^\infty \frac{[1+k^{2n-1}]^4[1-k^{2n-1}]^4}{[1-k^{2n}]^4[1+k^{2n}]^4}\sum_{n=1}^\infty\left[\frac{k^{2n-2}}{(1+k^{2n-1})^2} + \frac{k^{2n-2}}{(1-k^{2n-1})^2}\right] + \\ &+\frac{iN}{32\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\int_0^\infty \frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}. \end{aligned} \tag{B.48}$$

Notice that in the previous calculation we do not need to compute the quadratic term in f and g coming from the piece in front of the square bracket in eq. (2.5) because its coefficient is zero. Finally, by using the following identities, which can be proven with the help of the heat equation (A.1)

$$\sum_{n=1}^\infty \frac{k^{2n-1}}{(1-k^{2n-1})^2} = -k\frac{d}{dk}\left[\frac{1}{2}\log\prod_{n=1}^\infty(1-k^{2n}) + \log\prod_{n=1}^\infty(1-k^{2n-1})\right],$$

[5]The generators of SU(N) are normalized as: $\mathrm{Tr}[T^a\,T^b] = \frac{1}{2}\delta^{ab}$.

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$$\sum_{n=1}^{\infty} \frac{k^{2n-1}}{(1+k^{2n-1})^2} = k\frac{d}{dk}\left[\frac{1}{2}\log\prod_{n=1}^{\infty}(1-k^{2n}) + \log\prod_{n=1}^{\infty}(1+k^{2n-1})\right],$$
$$\sum_{n=1}^{\infty} \frac{k^{2n}}{(1+k^{2n})^2} = k\frac{d}{dk}\left[\frac{1}{2}\log\prod_{n=1}^{\infty}(1-k^{2n}) + \log\prod_{n=1}^{\infty}(1+k^{2n})\right], \qquad \text{(B.49)}$$

we get the following compact expression for eq. (B.48):

$$Z_g^o(F^2) = -\frac{N}{8\pi^2}\int d^4x \left[-\frac{1}{4}F^a_{\alpha\beta}F^{a\alpha\beta}\right]\int_0^\infty \frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}\left[\frac{f_3(k)f_4(k)}{f_1(k)f_2(k)}\right]^4 2k\frac{d}{dk}\log\left[\frac{f_3(k)}{f_4(k)}\right] + $$
$$+\frac{iN}{32\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\int_0^\infty \frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}, \qquad \text{(B.50)}$$

which reproduces eq. (2.12).

C. Closed string channel

In this appendix we derive eqs. (2.9) and (2.10), which added together provide the interaction amplitude between a stack of N D3 branes and a brane dressed with an external field. The two contributions correspond to the propagation of untwisted and twisted closed string states. They were computed in the appendix of ref. [25] in the case of vanishing external gauge field.

The boundary state describing a Dp-brane without any gauge field on its world-volume and living on the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ is given in ref. [25] and it is the sum of two terms, one relative to the untwisted sector and the other to the twisted one. The boundary state describing a dressed brane completely transverse to the orbifold is given, for the untwisted sector U, by:

$$|\mathrm{D}p;F\rangle^U =, \frac{T_p}{2\sqrt{2}}\left(|\mathrm{D}p;F\rangle^U_{\mathrm{NS}} + |\mathrm{D}p;F\rangle^U_{\mathrm{R}}\right) \qquad \text{(C.1)}$$

where $|\mathrm{D}p;F\rangle^U_{\mathrm{NS}}$ and $|\mathrm{D}p;F\rangle^U_{\mathrm{R}}$ are the usual boundary states for a dressed bulk Dp-brane given in refs. [26, 34].

In the twisted sector T, instead, the dressed boundary state is given by:

$$|\mathrm{D}p;F\rangle^T = -\frac{T_p}{2\sqrt{2}\,\pi^2\alpha'}\left(|\mathrm{D}p;F\rangle^T_{\mathrm{NS}} + |\mathrm{D}p;F\rangle^T_{\mathrm{R}}\right) \qquad \text{(C.2)}$$

where

$$|\mathrm{D}p;F\rangle^T_{\mathrm{NS,R}} = \frac{1}{2}\left(|\mathrm{D}p;F+\rangle^T_{\mathrm{NS,R}} + |\mathrm{D}p;F-\rangle^T_{\mathrm{NS,R}}\right), \qquad \text{(C.3)}$$

and the Ishibashi states $|\mathrm{D}p;F\,\eta=\pm\rangle^T_{\mathrm{NS,R}}$ are

$$|\mathrm{D}p;F\,\eta\rangle^T_{\mathrm{NS,R}} = |\mathrm{D}p_X;F\rangle^T|\mathrm{D}p_\psi;F\,\eta\rangle^T_{\mathrm{NS,R}} \qquad \text{(C.4)}$$

with[6]

$$|\mathrm{D}p_X;F\rangle^T = \sqrt{-\det(\eta+\hat{\mathrm{F}})}\,\delta^{(5-p)}(\hat{q}^i - y^i)\times\prod_{n=1}^{\infty}\left[\mathrm{e}^{-\frac{1}{n}\alpha^\alpha_{-n}M_{\alpha\beta}\tilde{\alpha}^\beta_{-n}}\mathrm{e}^{\frac{1}{n}\alpha^i_{-n}\tilde{\alpha}^i_{-n}}\right]\times$$

[6] In eq. (C.4) we omit the ghost and superghost contributions which are not affected by the orbifold projection.

$$\times \prod_{r=\frac{1}{2}}^{\infty} \mathrm{e}^{-\frac{1}{r}\alpha^{\ell}_{-r}\tilde{\alpha}^{\ell}_{-r}} \prod_{\beta} |p_\beta = 0\rangle \prod_i |p_i = 0\rangle \,,$$

$$|\mathrm{D}p_\psi; F\,\eta\rangle^T_{NS} = \prod_{r=\frac{1}{2}}^{\infty} \left[\mathrm{e}^{\mathrm{i}\eta\psi^{\alpha}_{-r}M_{\alpha\beta}\tilde{\psi}^{\beta}_{-r}}\mathrm{e}^{-\mathrm{i}\eta\psi^{i}_{-r}\tilde{\psi}^{i}_{-r}}\right] \prod_{n=1}^{\infty} e^{\mathrm{i}\eta\psi^{\ell}_{-n}\tilde{\psi}^{\ell}_{-n}} |\mathrm{D}p_\psi, \eta\rangle^{(0)\;T}_{\mathrm{NS}} \,,$$

$$|\mathrm{D}p_\psi; F\,\eta\rangle^T_{R} = \frac{1}{\sqrt{-\det(\eta+\hat{\mathrm{F}})}} \prod_{n=1}^{\infty} \left[\mathrm{e}^{\mathrm{i}\eta\psi^{\alpha}_{-n}M_{\alpha\beta}\tilde{\psi}^{\beta}_{-n}}\mathrm{e}^{-\mathrm{i}\eta\psi^{i}_{-n}\tilde{\psi}^{i}_{-n}}\right] \times$$

$$\times \prod_{r=\frac{1}{2}}^{\infty} \mathrm{e}^{\mathrm{i}\eta\psi^{\ell}_{-r}\tilde{\psi}^{\ell}_{-r}} |\mathrm{D}p_\psi; F\,\eta\rangle^{(0)\;T}_{\mathrm{R}} \,. \tag{C.5}$$

In these expressions the longitudinal indices α, β take the values $0, 1, \ldots p$, the transverse index i takes the values $p+1, \ldots, 5$, while the index ℓ labels the orbifold directions. Furthermore the matrix M is defined by:

$$M^{\alpha}_{\beta} = \left[(1-\hat{F})(1+\hat{F})^{-1}\right]^{\alpha}_{\beta} \,. \tag{C.6}$$

The zero-mode part of the boundary state in the NS-NS sector has the same structure as the one without gauge field, while in the R-R sector it reads as:

$$|\mathrm{D}p_\psi; F\,\eta\rangle^{(0)\;T}_{\mathrm{R}} = \left(C\gamma^0 \ldots \gamma^p \frac{1+i\eta_1\tilde{\gamma}}{1+i\eta_1} ; e^{-\frac{1}{2}\hat{F}_{\alpha\beta}\gamma^\alpha\gamma^\beta} ;\right)_{ab} |a\rangle|\widetilde{b}\rangle|D_{\mathrm{sgh}}, \eta_1\rangle^{(0)}_{\mathrm{R}} \tag{C.7}$$

where the symbol ; ; means that we have to expand the exponential and then to antisymmetrize the indices of the γ matrices. The superghost zero mode contribution is unchanged with respect to the untwisted sector and can be found in ref. [26]. Here the γ matrices reproduce the Clifford algebra in six dimensions and $\tilde{\gamma} \equiv \prod_{i=0}^{5} \gamma^i$.

We have now introduced all the ingredients necessary to compute the tree level closed string amplitude given in eqs. (2.7) and (2.8), that are relative to the case $p = 3$ we are interested in. The explicit expression for the untwisted component Z^c_e is:

$$Z^c_e = \frac{1}{4}\frac{1}{(8\pi^2\alpha')^2} \int d^4x \sqrt{-\det(\eta+\hat{F})} \int_0^{+\infty} \frac{dt}{t^3} \frac{e^{-\frac{y^2}{2\pi\alpha' t}}}{\prod_{n=1}^{+\infty}\det\left(1-M^T q^{2n}\right)\left(1-q^{2n}\right)^4} \times$$

$$\times\left\{\frac{1}{q}\left[\prod_{n=1}^{+\infty}\det(1+M^T q^{2n-1})\left(1+q^{2n-1}\right)^4 - \prod_{n=1}^{+\infty}\det\left(1-M^T q^{2n-1}\right)\left(1-q^{2n-1}\right)^4\right] - \right.$$

$$\left. - \frac{2^4}{\sqrt{-\det(\eta+\hat{F})}} \prod_{n=1}^{+\infty}\det(1+M^T q^{2n})\left(1+q^{2n}\right)^4\right\} , \tag{C.8}$$

where $q = e^{-\pi t}$. We are not going to derive in detail eq. (C.8), but we want to point out that it can be easily obtained from eq. (A.2) of ref. [25], rewritten in the closed string channel by adding in the NS-NS sector the Born-Infeld action and the contribution of the gauge fields in the brane world-volume directions. Furthermore we need to make the following substitutions:

$$\prod_{n=1}^{\infty}(1 \pm q^{2n-1})^4 \to \prod_{n=1}^{\infty} \det(1 \pm M^T q^{2n-1}) \tag{C.9}$$

and

$$\prod_{n=1}^{\infty}(1 \pm q^{2n})^4 \to \prod_{n=1}^{\infty} \det(1 \pm M^T q^{2n}) \,. \tag{C.10}$$

Analogous substitutions have to be done in the twisted sector. In particular, in the NS-NS sector the expression for the tree level closed string amplitude given in eq. (2.8) is equal to:

$$\begin{aligned} Z_h^{c(\mathrm{NS-NS})} &= \frac{4}{(8\pi^2\alpha')^2} \int d^4x \sqrt{-\det(\eta + \hat{F})} \int_0^{+\infty} \frac{dt}{t} e^{-\frac{y^2}{2\pi\alpha' t}} \times \\ &\times \prod_{n=1}^{+\infty} \frac{\det(1 + M^T q^{2n-1}) \left(1 + q^{2n}\right)^4}{\det(1 - M^T q^{2n})(1 - q^{2n-1})^4} \,. \end{aligned} \tag{C.11}$$

The zero modes in the NS-NS twisted sector coincide with those with vanishing gauge field. Hence, as shown in ref. [25], the spin structure NS-NS$(-1)^F$ does not contribute to the interaction in this sector.

In the R-R sector we can proceed in an analogous way, but in this case we have also a contribution from the zero modes that is divergent and requires to be treated more carefully through a suitable regularization. Let us evaluate it explicitly. According to ref. [32] we insert the regulator $\mathcal{R}(x) = x^{2(F_0+G_0)}$ as follows:

$$\begin{aligned} {}_{\mathrm{R}}^{T\,(0)}\langle \mathrm{D3}_\psi, \eta_2 | \mathrm{D3}_\psi; F\,\eta_1 \rangle_{\mathrm{R}}^{(0)} &\equiv \lim_{x\to 1} {}_{\mathrm{R}}^{T\,(0)}\langle \mathrm{D3}_\psi, \eta_2 |\, \mathcal{R}(x)\, | \mathrm{D3}_\psi; F\,\eta_1 \rangle_{\mathrm{R}}^{(0)\;T} \\ &= \lim_{x\to 1} \Big[{}_{\mathrm{R}}^{T\,(0)}\langle \mathrm{D3}_\psi, \eta_2 |\, x^{2F_0}\, | \mathrm{D3}_\psi; F \eta_1 \rangle_{\mathrm{R}}^{(0)\;T} \times \\ &\qquad \times\; {}_{\mathrm{R}}^{(0)}\langle \mathrm{D}_{\mathrm{sgh}}, \eta_2 |\, x^{2G_0}\, | \mathrm{D}_{\mathrm{sgh}}, \eta_1 \rangle_{\mathrm{R}}^{(0)} \Big]. \end{aligned} \tag{C.12}$$

where $(-1)^{F_0} = i\tilde{\gamma}$ and $G_0 = -\gamma_0\beta_0$.

For the superghost part the regularization scheme leads to the same result as in the untwisted sector [32, 26]:

$$\begin{aligned} {}_{\mathrm{R}}^{(0)}\langle B_{\mathrm{sgh}}, \eta_2 |\, x^{2G_0}\, | B_{\mathrm{sgh}}, \eta_1 \rangle_{\mathrm{R}}^{(0)} &= \langle -3/2, -1/2 | \mathrm{e}^{-i\eta_2\beta_0\widetilde{\gamma}_0}\, x^{-2\gamma_0\beta_0}\, \mathrm{e}^{i\eta_1\gamma_0\widetilde{\beta}_0} | -1/2, -3/2 \rangle \\ &= \frac{1}{1 + \eta_1\eta_2 x^2} \,. \end{aligned} \tag{C.13}$$

In order to evaluate the matter part we introduce the operators

$$N_1 \equiv \gamma^0\gamma^1 \qquad N_2 \equiv -i\gamma^2\gamma^3 \,; \qquad N_3 \equiv -i\gamma^4\gamma^5 \tag{C.14}$$

and write

$$(-1)^{F_0} = i\tilde{\gamma} = -i \prod_{k=1}^{3} N_k = \prod_{k=1}^{3} \exp\left(i\, N_k\, \pi/2\right) = (-1)^{\frac{1}{2}(N_1+N_2+N_3)} \,, \tag{C.15}$$

where we have used that $\exp\left(i\, N_k\, \pi/2\right) = i N_k$. From the previous equation we can read $F_0 = \frac{1}{2}\sum_{k=1}^{3} N_k$ and thus the regulator for the fermionic zero-modes can be written as follows:

$$x^{2F_0} = x^{\sum\limits_{k=1}^{3} N_k} . \tag{C.16}$$

Substituting this expression in the matter part of eq. (C.12) we get:

$$\begin{aligned} {}_{\rm R}^{T\,(0)}\langle {\rm D3}_\psi, \eta_2 | x^{2F_0} | {\rm D3}_\psi ; F\, \eta_1 \rangle_{\rm R}^{(0)\,T} &= -\delta_{\eta_1\eta_2;1} Tr[x^{2F_0}] + \\ &\quad + \delta_{\eta_1\eta_2;-1} (\pi\alpha')^2 F_{\alpha\beta} \tilde{F}^{\alpha\beta}\ {\rm Tr}\left[x^{2F_0} \tilde{\gamma}\gamma^0 \ldots \gamma^3\right] \end{aligned} \tag{C.17}$$

where we have only kept those terms which, added to the ghost contribution in eq. (C.13), yields a non-zero result. The traces appearing in the previous equation are easily evaluated

$$ {\rm Tr}[x^{2F_0}] = \prod_{k=1}^{3} {\rm Tr}[x^{N_k}] = \left(x + \frac{1}{x}\right)^3 , \tag{C.18}$$

$$ {\rm Tr}[x^{2F_0}\gamma^0\gamma^1\gamma^2\gamma^3] = i {\rm Tr}[x^{N_1} N_1] {\rm Tr}[x^{N_2} N_2] {\rm Tr}[x^{N_3}] = i \left(x - \frac{1}{x}\right)^2 \left(x + \frac{1}{x}\right) , \tag{C.19}$$

$$ {\rm Tr}[x^{2F_0}\gamma^0\gamma^1\gamma^2\gamma^3\tilde{\gamma}] = -i {\rm Tr}[x^{N_1}] {\rm Tr}[x^{N_2}] {\rm Tr}[x^{N_3} N_3] = -i \left(x + \frac{1}{x}\right)^2 \left(x - \frac{1}{x}\right) . \tag{C.20}$$

By plugging them in eq. (C.17), including also the ghost contribution and performing the $x \to 1$ limit one gets:

$$ {}_{\rm R}^{T\,(0)}\langle {\rm D3}_\psi, \eta_2 | {\rm D3}_\psi ; F\, \eta_1 \rangle_{\rm R}^{(0)\,T} = -4\delta_{\eta_1\eta_2;1} + 4i\delta_{\eta_1\eta_2;-1} \left(\frac{(\pi\alpha')^2}{2} F^a_{\alpha\beta} \tilde{F}^{a\alpha\beta} \right) \tag{C.21}$$

where an extra factor $1/2$ has been introduced to take in account the trace over the SU(N) generators. By adding to the previous expression the contribution of the non zero modes it is straightforward to write down the complete expression for the interaction in the twisted R-R sector:

$$\begin{aligned} Z_h^{c({\rm R-R})} &= -\frac{4}{(8\pi^2\alpha')^2} \int d^4x \sqrt{-\det(\eta + \hat{F})} \times \\ &\quad \times \int_0^{+\infty} \frac{dt}{t} \frac{e^{-\frac{y^2}{2\pi\alpha' t}}}{\prod_{n=1}^{+\infty} \det(1 - M^T q^{2n})(1 - q^{2n-1})^4} \times \\ &\quad \times \frac{\prod_{n=1}^{+\infty} \det\left(1 + M^T q^{2n}\right)\left(1 + q^{2n-1}\right)^4}{\sqrt{-\det(\eta + \hat{F})}} + \frac{iN}{32\pi^2} \int d^4x F^a_{\alpha\beta} \tilde{F}^{a\alpha\beta} \int_0^\infty \frac{dt}{t} e^{-\frac{y^2}{2\pi\alpha' t}} . \end{aligned} \tag{C.22}$$

The determinants of the terms containing the external gauge field present in eqs. (C.11) and (C.22) can be computed using the parametrization for F given in eq. (2.2) and one gets the following expressions in terms of the Θ functions defined in appendix A:

$$\prod_{n=1}^{\infty} \det\left(1 + M^T q^{2n-1}\right) = \Theta_3(\nu_f | it)\Theta_3(\nu_g | it) \prod_{n=1}^{\infty} (1 - q^{2n})^{-2} , \tag{C.23}$$

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$$\prod_{n=1}^{\infty} \det\left(1 - M^T q^{2n}\right) = e^{\pi t/2} \frac{\Theta_1(\nu_f|it)}{2\sin\pi\nu_f} \frac{\Theta_1(\nu_g|it)}{2\sin\pi\nu_g} \prod_{n=1}^{\infty} (1-q^{2n})^{-2}, \tag{C.24}$$

$$\prod_{n=1}^{\infty} \det\left(1 + M^T q^{2n}\right) = e^{\pi t/2} \frac{\Theta_2(\nu_f|it)}{2\cos\pi\nu_f} \frac{\Theta_2(\nu_g|it)}{2\cos\pi\nu_g} \prod_{n=1}^{\infty} (1-q^{2n})^{-2}, \tag{C.25}$$

$$\prod_{n=1}^{\infty} \det\left(1 - M^T q^{2n-1}\right) = \Theta_4(\nu_f|it)\Theta_4(\nu_g|it) \prod_{n=1}^{\infty} (1-q^{2n})^{-2}. \tag{C.26}$$

Inserting these formulas in eqs. (C.8), (C.11), (C.22) and using the definition of the functions f_i given in appendix B, one easily gets eqs. (2.9) and (2.10).

In the last part of this appendix we expand eq. (2.10) keeping up to terms quadratic in the gauge field. This can be done by using the following expansions:

$$\Theta_3(\nu_f|it) = \Theta_3(0|it)\left[1 + 4f^2 \sum_{n=1}^{\infty} \frac{q^{2n-1}}{(1+q^{2n-1})^2} + \ldots\right], \tag{C.27}$$

$$\Theta_2(\nu_f|it) = \Theta_2(0|it)\left[1 + 4f^2 \sum_{n=1}^{\infty} \frac{q^{2n}}{(1+q^{2n})^2} + \frac{1}{2}f^2 + \ldots\right] \tag{C.28}$$

and the analogous ones for $\nu_f \to \nu_g$ together with the following equation:

$$\lim_{\nu\to 0} \frac{2\sin\pi\nu}{\Theta_1(\nu|it)} = -\frac{1}{f_1^3(q)}. \tag{C.29}$$

Notice that the factor in front of the bracket in eq. (2.10) can just be computed for $f = g = 0$, because the corresponding quadratic term in f and g has vanishing coefficient as a consequence of the fact that the annulus diagram for two undressed fractional branes vanishes. One can then use eq. (B.47) for rewriting the combination $f^2 - g^2$ in terms of the kinetic term of the gauge field and, after some calculation, one obtains the following expression for the quadratic terms of the gauge field:

$$\begin{aligned} Z_h^c(F^2) &= \frac{N}{8\pi^2} \int d^4x \left(-\frac{1}{4} F^a_{\alpha\beta} F^{a\alpha\beta}\right) \int_0^{\infty} \frac{dt}{t} e^{-\frac{y^2}{2\pi\alpha' t}} \prod_{n=1}^{\infty} \frac{[1+q^{2n-1}]^4 [1+q^{2n}]^4}{[1-q^{2n-1}]^4 [1-q^{2n}]^4} \times \\ &\times \left[-1 + 8 \sum_{n=1}^{\infty} \left(\frac{q^{2n-1}}{[1+q^{2n-1}]^2} - \frac{q^{2n}}{[1+q^{2n}]^2}\right)\right] + \\ &+ \frac{iN}{32\pi^2} \int d^4x F^a_{\alpha\beta} \tilde{F}^{a\alpha\beta} \int_0^{\infty} \frac{dt}{t} e^{-\frac{y^2}{2\pi\alpha' t}}. \end{aligned} \tag{C.30}$$

By using the identities in eq. (B.49), eq. (C.30) becomes:

$$\begin{aligned} Z_h^c(F^2) &= \frac{N}{8\pi^2} \int d^4x \left(-\frac{1}{4} F^a_{\alpha\beta} F^{a\alpha\beta}\right) \int_0^{\infty} \frac{dt}{t} \mathrm{e}^{-\frac{y^2}{2\pi\alpha' t}} \left[\frac{f_3(q) f_2(q)}{f_1(q) f_4(q)}\right]^4 2q \frac{d}{dq} \log\left[\frac{f_3(q)}{f_2(q)}\right] + \\ &+ \frac{iN}{32\pi^2} \int d^4x F^a_{\alpha\beta} \tilde{F}^{a\alpha\beta} \int_0^{\infty} \frac{dt}{t} e^{-\frac{y^2}{2\pi\alpha' t}}, \qquad q = e^{-\pi t} \end{aligned} \tag{C.31}$$

that reproduces eq. (2.13). Using the modular transformations of the f-functions and the relation

$$q \frac{d}{dq} = -\tau^2 k \frac{d}{dk} \tag{C.32}$$

one can easily check that the previous equation is properly mapped in eq. (B.50) evaluated in the open channel by the open/closed string duality.

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